2. One-dimensional formulation

We reformulate $(*_s)$ as a nonlinear equation in the form

$$F(\lambda, w)=0, \text{ where } w: \mathbb{R} \to \mathbb{R} \text{ is } 2\pi\text{-periodic}.$$

Let ϕ be such that $\phi + i\psi$ is <u>holomorphic</u> in Ω, i.e. $\begin{cases} \phi_x = \psi_y \\ \phi_y = -\psi_x \end{cases}$. Cauchy-Riemann eqn.

complex differentible at every point $x \in \Omega$

⌜ holomorphic $\Leftrightarrow$ complex analytic, i.e. infinitely differentible and $f(x) = T(x)$ $\nwarrow$ Taylor series ⌟

$$\left.\begin{array}{l} \psi \in C^{2,\alpha}(\bar{\Omega}) \\ \text{Cauchy-Riemann} \end{array}\right\} \Rightarrow \phi \in C^{2,\alpha}(\bar{\Omega})$$

Moreover,

(a) $\phi(-x,y) = -\phi(x,y) \quad \forall_{(x,y)\in\Omega}$

⌜ $(*_s)$ ④ $\Rightarrow \psi(-x,y) = \psi(x,y) \Rightarrow \psi_y(-x,y) = \psi_y(x,y) \overset{C\text{-}R}{\Rightarrow} \phi_x(-x,y) = \phi_x(x,y)$, i.e. ϕ_x is even in x

$\overset{\phi(0,0)=0}{\Rightarrow} \phi$ is odd in x ⌟

(b) $\phi_y(x,y) = 0 \quad \forall_{\substack{x=0,\pm\pi, \\ (x,y)\in\bar{\Omega}}}$ i.e. $\phi(0,y)$ and $\phi(\pm\pi,y)$ are independent of y

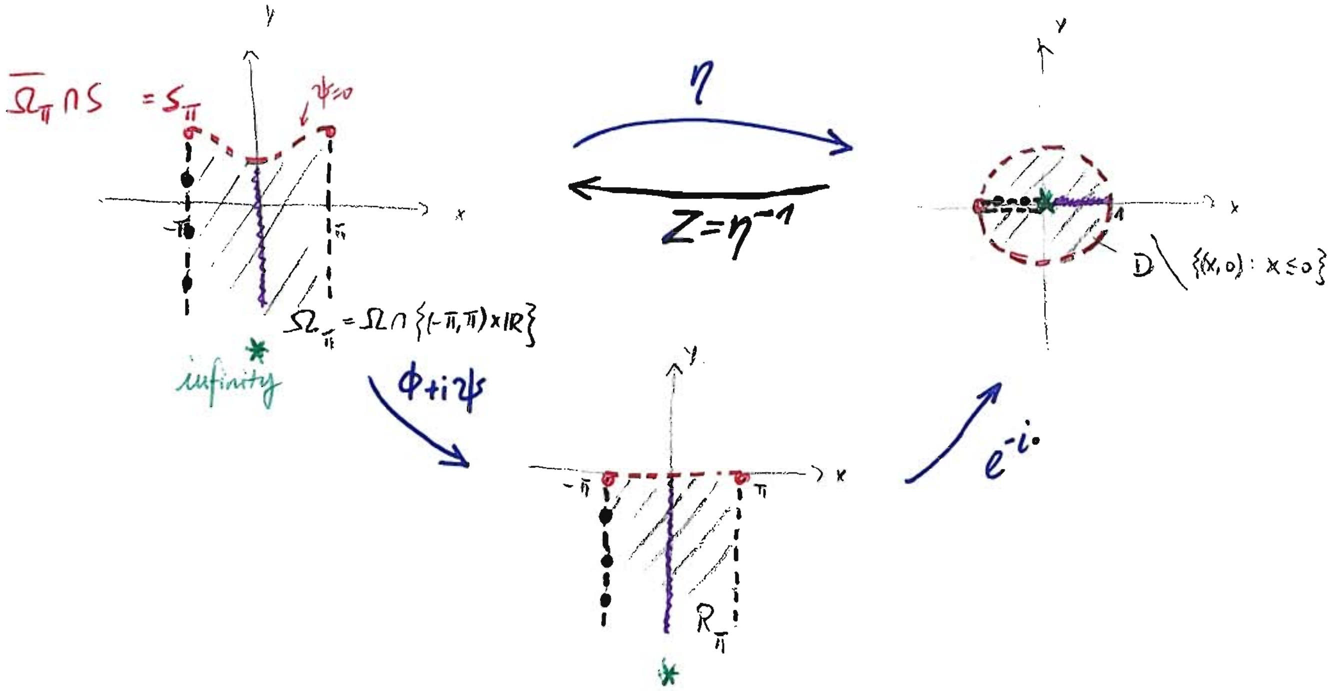
⌜ ψ is 2π-per. and even in x $\Rightarrow \psi_x(x,y) = 0 \quad \forall_{x=0,\pm\pi\, (x,y)\in\Omega} \overset{C\text{-}R}{\Longrightarrow} \phi_y(x,y) = 0 \quad \forall_{x=0,\pm\pi,\,(x,y)\in\Omega}$. ⌟

(c) $\phi(\pm\pi, y) = \pm\pi, \quad \phi(0,y) = 0 \quad \forall_y$

⌜ $(*_s)$ ③ $\Rightarrow \underset{\text{as } y\to-\infty}{(\psi_x, \psi_y) \to (0,1)} \Rightarrow \phi(\pi,y) - \phi(-\pi,y) = \int_{-\pi}^{\pi} \phi_x(x,y)dx \overset{C\text{-}R}{=} \int_{-\pi}^{\pi} \underbrace{\psi_y(x,y)}_{\to 1} dx \to 2\pi$ as $y \to -\infty$

$$\left.\begin{array}{l} (a) \Rightarrow \phi(-\pi,y) = -\phi(\pi,y) \\ (b) \Rightarrow \phi(\pm\pi,y) \text{ is independent of } y \Rightarrow \phi(\pi,y) - \phi(-\pi,y) = 2\pi \end{array}\right] \Rightarrow \phi(\pi,y) = \pi, \ \phi(-\pi,y) = -\pi$$

$\phi(0,y)$ is independent of $y \Rightarrow \phi(0,y) = \phi(0,0) = 0$ $\nwarrow$ ϕ is odd in x ⌟



(d) $\phi+i\psi : \Omega_\pi \to R_\pi$ is a <u>conformal</u> bijection

i.e. preserves angles ⟺ the Jacobian derivative matrix is a scalar times rotation matrix

$$C\text{-}R \Rightarrow \begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (a^2+b^2)\cdot \underbrace{\begin{pmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix}}_{\text{a rotation matrix } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

thus, $\eta : \Omega_\pi \to D\setminus\{(x,0): x\le 0\}$ is a conformal bijection.

(e) let Z be the inverse of η. Then

$$Z(\xi) - i\underbrace{\log\xi}_{\text{the branch s.t. } \log\xi\in\mathbb{R} \text{ for } \xi\in\mathbb{R}^+} \text{ for } \xi\in D\setminus\{(x,0): x\le 0\} \xrightarrow{\text{extension}} Z(\xi) - i\log\xi, \text{ for } \xi\in D$$

and $Z(\cdot) - i\log\cdot$ is $\mathbb{C}$-analytic in D.

let $\xi = re^{i\pi}$, then $Z(\xi) = -\pi + iy$, $i\log\xi = i\log r - \pi$

$\xi = re^{-i\pi}$, then $Z(\xi) = \pi + iy$, $i\log\xi = i\log r + \pi$

$$Z(\cdot) - i\log\cdot\Big|_{re^{i\pi}} = Z(\cdot) - i\log\cdot\Big|_{re^{-i\pi}}$$

Thus, $Z(\xi) = i\left(\log\xi + \sum_{k=0}^{\infty} a_k \xi^k\right) = i(\log\xi + f(\xi))$ for a $\mathbb{C}$analytic function f in D.

Also, $\underbrace{f(\xi)\in\mathbb{R} \text{ if } \xi\in\mathbb{R}}$, thus all coefficients $a_k\in\mathbb{R}$.

$f(\xi)\in\mathbb{R} \Leftrightarrow Z(\xi) - i\log\xi \in i\mathbb{R}$ ✓

if $\xi\in\mathbb{R}^+$ then $Z(\xi)\in i\mathbb{R}$, $i\log\xi \in i\mathbb{R}$

if $\xi\in\mathbb{R}^-$, then $Z(\xi) - i\log(\xi) \in i\mathbb{R}$

Write $\xi = re^{it} \in D$ (t no longer denotes time). Then,

$$Z(\xi) = i\Big(\log r + it + \sum_{k=0}^{\infty} a_k r^k \underbrace{e^{ikt}}_{\cos kt + i\sin kt}\Big)$$

$$= \underbrace{-t - \sum_{k=0}^{\infty} a_k r^k \sin(kt)}_{x} + i\underbrace{\Big(\log r + \sum_{k=0}^{\infty} a_k r^k \cos(kt)\Big)}_{y} \in \Omega_\pi, \qquad (P)$$

— this is a parametrization of $(x,y) \in \Omega_\pi$ using (r,t).

where S_π corresponds to $\partial D = \{\xi : r = 1\}$, i.e.

$$S_\pi = \{(x,y) \in \Omega_\pi : x = -t - \sum a_k \sin(kt),\ y = \sum a_k \cos(kt),\ t \in [-\pi, \pi]\} \quad (S)$$

$\psi \to \phi + i\psi \to Z \to f$

Now let us reformulate the above procedure by starting with f (forgetting ϕ and ψ). Suppose f to be a holomorphic function on $\overline{D}$ (where $f(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$ for $a_k \in \mathbb{R}$, $\forall_{k=0,1,\dots}$ (thus $f(\xi) \in \mathbb{R}$ for $\xi \in \mathbb{R}$)) Let Z be defined by

$$(Z) \qquad Z(\xi) = i(\log \xi + f(\xi)), \quad \xi \in D. \qquad \text{(or equivalently, } Z \text{ be defined by (P))}$$

Assume that Z is a bijection onto its range Ω_π (then (P) gives a parametrization of Ω_π). Let S_π be defined by (S) (i.e. $S_\pi = Z(e^{it}) = Z(r{=}1)$)

Define $\Phi + i\Psi$ on Ω_π (using (P)) by

$$(\Psi) \qquad \Phi(\underbrace{x+iy}_{Z(\xi)}) + i\Psi(\underbrace{x+iy}_{Z(\xi)}) = i\log \underset{\substack{\uparrow \\ Z^{-1}(x+iy)}}{\xi} \overset{\xi = re^{it}}{=} i\log r - t, \quad 0 \le r \le 1,\ -\pi \le t \le \pi.$$

? does $\Psi = \Psi(x,y)$ define a solution of $(*_S)$

$(*_s)$ ① $\Delta\Psi = 0$ in Ω_π

this is due to that $\Phi + i\Psi$ is holomorphic $\overset{C-R}{\Rightarrow}$ Φ, Ψ are both harmonic

$(*_s)$ ② $\Psi = 0$ on S_π on S_π, we have $r=1$, thus by definition, $\Psi = \log r = \log 1 = 0$.

$(*_s)$ ③ $\nabla\Phi(x,y) \to (0,1)$ as $y \to -\infty$

$$\left.\begin{array}{l} y\to-\infty \Rightarrow r\to 0 \overset{(P)}{\Longrightarrow} x\to -t \\ \Phi(x,y) = -t \end{array}\right\} \Rightarrow \underset{\substack{\text{as } r\to 0 \\ (\text{as } y\to-\infty)}}{\Phi(x,y)\to x} \Rightarrow \nabla\Phi(x,y)\to(1,0) \text{ as } y\to-\infty$$

$$\overset{C-R}{\Longrightarrow} \nabla\Psi(x,y)\to(0,1) \text{ as } y\to-\infty.$$

$(*_s)$ ④ $\Psi(-x,y) = \Psi(x,y)$ $\underset{(x,y)\in\Omega_\pi}{\forall}$

$$(P) \Rightarrow \begin{cases} x = -t - \sum a_k r^k \sin(kt) \\ y = \log r + \sum a_k r^k \cos(kt) \end{cases} \Rightarrow \text{if } (r,t) \text{ gives } (x,y) \text{ then } (r,-t) \text{ gives } (-x,y) \Rightarrow (\pm x, y) \text{ corresponds to the same } r$$

$$\xrightarrow{\Psi(x,y)=\log r} (\pm x, y) \text{ gives the same } \Psi, \text{ i.e. } \Psi(x,y) = \Psi(-x,y).$$

$(*_s)$ ⑤ $\frac{1}{2}|\nabla\Psi(x,y)|^2 + \lambda y \equiv \frac{1}{2}$ on S_π

We rewrite this condition using a_k's (the coefficients of f).

$$(\Phi+i\Psi)(Z(\xi)) = i\log\xi \Rightarrow \underbrace{\frac{d}{dz}(\Phi+i\Psi)}_{=\Phi_x - i\Phi_y = \Psi_y + i\Psi_x}(Z(\xi))\frac{dZ}{d\xi} = \frac{i}{\xi}$$

$$|\nabla\Psi(Z(\xi))|^2 \overset{\swarrow \det J_{\mathbb{R}}}{=} \Psi_x(Z(\xi))^2 + \Psi_y(Z(\xi))^2 = \left|\frac{d}{dz}(\Phi+i\Psi)(Z(\xi))\right|^2 \overset{\swarrow |\det J_{\mathbb{C}}|^2}{=} \frac{1}{|\xi|^2|Z'(\xi)|^2}$$

$$\left.\begin{array}{l} \text{when } (x,y)\in S_\pi \Rightarrow |\xi| = r = 1 \Rightarrow |\nabla\Psi(Z(\xi))|^2 = \frac{1}{|Z'(\xi)|^2} \\ Z'(\xi) = \frac{i}{\xi} + i\sum_{k=1}^{\infty} k a_k \xi^{k-1} = \frac{i}{\xi}\left(1 + \sum_{k=1}^{\infty} k a_k \xi^k\right) \end{array}\right\} \Rightarrow$$

on S_π:
$$|\nabla\Psi(Z(\xi))|^2 = \frac{1}{\left|1 + \sum_{k=1}^{\infty} k a_k \xi^k\right|^2} = \frac{1}{\left(1+\sum_{k=1}^{\infty} k a_k \cos kt\right)^2 + \left(\sum_{k=1}^{\infty} k a_k \sin kt\right)^2}$$

for convenience, define an operator $\mathcal{C} : L^2_{[-\pi,\pi]} \to L^2_{[-\pi,\pi]}$ by

$$\mathcal{C}(1)=0, \quad \mathcal{C}(\sin kt) = -\cos kt, \quad \mathcal{C}(\cos kt) = \sin kt, \qquad k \geq 1,$$

extended by linearity and continuity (Riesz-Fischer Thm: a measurable function on $[-\pi,\pi]$ is square integrable $\Leftrightarrow$ its Fourier series coverges in L^2-norm),

which is a $\underbrace{\text{bounded}}_{\|\mathcal{C}\| \leq 1}$ linear operator.

Let w be defined by $w(t) = \sum_{k=0}^{\infty} a_k \cos kt$ for $t \in [-\pi,\pi]$, where a_k's are coefficients of f, ($\Rightarrow w \in L^2$, $w(-t)=w(t)$)

such that $w' \in L^2[-\pi,\pi]$. Then, $w(-\pi)=w(\pi)$ and

$$1 + \mathcal{C} w'(t) = 1 + \sum_{k=1}^{\infty} k a_k \cos kt .$$

Thus, $|\nabla \Phi(Z(\xi))|^2$ can be rewritten as

$$|\nabla \Phi(Z(\xi))|^2 = \frac{1}{(1+\mathcal{C} w'(t))^2 + w'(t)^2}$$

and (S) becomes

$$S_\pi = \{ (-t - \mathcal{C} w(t), w(t)) : t \in [-\pi,\pi] \}$$

$$= \{ (t + \mathcal{C} w(t), w(t)) : t \in [-\pi,\pi] \} \qquad \text{since } w(-t)=w(t).$$

Therefore, "$(*_s)$ ⑤ $\frac{1}{2} |\nabla \Phi(x,y)|^2 + \lambda y \equiv \frac{1}{2}$ on S_π" is equivalent to

$$\text{(CP)} \qquad (1 - 2\lambda w(t))(w'(t)^2 + (1+\mathcal{C} w'(t))^2) = 1 \qquad \text{for almost all } t \in [-\pi,\pi].$$

"constant pressure" $\nearrow$

In summary, if $w \in C^{2,\alpha}$ is even (thus $w(t) = \sum_{k=0}^{\infty} a_k \cos kt$) and such that Z defined by (Z) (using a_k's) is globally injective (thus bijetive on its range) and such that (CP) holds, then there exists a symmetric Stokes wave with profile : $\{ (t + \mathcal{C} w(t), w(t)) : t \in \mathbb{R} \}$.

• Conjugation operator

$\mathcal{C} : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$, $\mathcal{C}(1)=0$, $\mathcal{C}(\sin kt) = -\cos kt$, $\mathcal{C}(\cos kt) = \sin kt$ $\forall k \geq 1$

a. complex analysis :

In complex notation ($e^{int} = \cos(nt) + i\sin(nt)$), we have

$$\mathcal{C}\, e^{int} = -i \,\mathrm{sign}(n)\, e^{int}, \quad n \in \mathbb{Z},$$

with the convention $\mathrm{sign}(n) = \begin{cases} 1 & \text{if } n>0 \\ 0 & \text{if } n=0 \\ -1 & \text{if } n<0 \end{cases}$.

Let u be a 2π-periodic, smooth, real-valued function with

$$u(t) = \sum_{n\in\mathbb{Z}} \underbrace{\hat{u}(n)}_{\text{the Fourier coefficient} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\, e^{-int}\, dt} e^{int}, \quad t \in [-\pi,\pi]$$

$\ulcorner\, u(t) \in \mathbb{R} \Rightarrow u(t) = \overline{u(t)} \Rightarrow \underline{\hat{u}(-n) = \overline{\hat{u}(n)}} \,\lrcorner$

$$= \hat{u}(0) + \sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int} + \overline{\sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int}}$$

$$= \hat{u}(0) + 2\, \underbrace{\mathrm{Re}}_{\text{the real part of a complex number}} \Big(\sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int} \Big)$$

Based on the coefficients $\hat{u}(n)$'s, we define a complex holomorphic function $f : \underset{\{z\in\mathbb{C} :\ |z|\leq 1\}}{\overline{D}} \to \mathbb{C}$ by

$$f(z) = \hat{u}(0) + 2 \sum_{n\in\mathbb{N}} \hat{u}(n)\, z^n, \quad z\in\mathbb{C},\ |z|\leq 1$$

$$:= U(z) + i\,V(z), \quad \text{i.e. } U(z) = \mathrm{Re}\, f(z),\ V(z) = \mathrm{Im}\, f(z).$$

where U and V are real, and $V(0)=0$ (since $f(0) = \hat{u}(0) \in \mathbb{R}$).

Note that $\boxed{u(t) = U(e^{it})}$ by definition of U, and

$$\boxed{V(e^{it})} = \mathrm{Im} f(e^{it}) \overset{\hat{u}(0)\in\mathbb{R}}{=} \mathrm{Im}\Big(2 \underbrace{\sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int}}_{\alpha + i\beta} \Big) = 2\beta = -i\big(\alpha + i\beta - \overline{\alpha + i\beta}\big)$$

$$= -i \Big(\sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int} - \overline{\sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int}} \Big) = -i \Big(\sum_{n\in\mathbb{N}} \hat{u}(n)\, e^{int} - \sum_{n\in\mathbb{N}} \hat{u}(-n)\, e^{-int} \Big)$$

$$= -i \cdot \sum_{n\in\mathbb{Z}} \mathrm{sign}(n)\, \hat{u}(n)\, e^{int} = \boxed{\mathcal{C} u(t)}.$$

That is, a 2π-periodic smooth real-valued function u can be viewed as the real part U of a complex holomorphic function f (defined on $\bar{D}$) when restricted to $\underset{\text{the unit circle}}{S^1 := \partial\bar{D}}$, and the corresponding imaginary part V is precisely given by $\mathcal{C}u$ (when restricted to S^1).

Moreover, we have

$$(*_U) \qquad \mathcal{C}u'(t) = (\mathcal{C}u(t))' = \frac{\partial V}{\partial t}\Big|_{e^{it}} = \frac{\partial U}{\partial r}\Big|_{e^{it}}$$

$$V = V(x,y) = V(r\cos t, r\sin t)$$

$$\Rightarrow V_t = V_x(-r\sin t) + V_y r\cos t$$

$$\Rightarrow V_t\Big|_{e^{it}} = V_t\Big|_{r=1} = -V_x \sin t + V_y \cos t$$

$$\overset{C\text{-}R}{=} U_y \sin t + U_x \cos t$$

$$= U_r$$

where $\underbrace{\Delta U = 0}_{\text{since } U \text{ is the real part of a holomorphic function}}$ on D, $U(e^{it}) = u(t)$.

b. Functional analysis cf. Zygmund, "Trigonometric series, I"

for $u \in L^2[-\pi,\pi]$, $\mathcal{C}u$ is given pointwise almost everywhere by a singular integral: $\mathcal{C}u(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(s)ds}{\tan\frac{t-s}{2}} = \lim_{\varepsilon\to 0+}\left(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi}\right)$

$\underline{\text{Theorem}}^{10.2.1}$ (M. Riesz) $\mathcal{C}: L^p[-\pi,\pi] \to L^p[-\pi,\pi]$ is a bounded linear operator for all $p\in(1,\infty)$.

$\underline{\text{Theorem}}^{10.2.2.}$ (Privalov) $\mathcal{C}: C^\alpha \to C^\alpha$ is a bounded linear operator for all $\alpha\in(0,1)$.

Theorem 10.2.3 Let $u \in L^2[-\pi,\pi]$. Then there exists a holomorphic function f defined on the unit disc such that, in $L^2[-\pi,\pi]$ and pointwise for almost all $t \in [-\pi,\pi]$,

$$\lim_{r \to 1} f(re^{it}) = u(t) + i\,\mathcal{C}u(t).$$

Let $u, w \in L^2[-\pi,\pi]$, $\alpha, \beta \in \mathbb{R}$ and f, g are holomorphic in the unit disc with

$$\lim_{r\to 1} f(re^{it}) = u(t) + i(\alpha + \mathcal{C}u(t)), \quad \lim_{r\to 1} g(re^{it}) = w(t) + i(\beta + \mathcal{C}w(t))$$

pointwise almost everywhere for $t \in [-\pi,\pi]$, in $L^2[-\pi,\pi]$. Then, there exists $\gamma \in \mathbb{R}$ and $v \in L^1[-\pi,\pi]$ such that $\mathcal{C}v \in L^1[-\pi,\pi]$ and in $L^1[-\pi,\pi]$ and pointwise almost everywhere for $t \in [-\pi,\pi]$,

$$\lim_{r\to 1} f(re^{it})\, g(re^{it}) = v(t) + i(\gamma + \mathcal{C}v(t)).$$

In particular, if $fg|_{S^1}$ is imaginary, then $v=0$ and $fg = i\gamma$ on D for some real constant γ.

Let u be even and 2π-periodic. Let $\underbrace{\hat{u}(t)}_{\uparrow \text{the } \pi\text{-shift}} = u(t+\pi)$. Then, $\hat{u}$ is even, 2π-periodic, and

$$\mathcal{C}\hat{u}(t) = (\mathcal{C}u)(t+\pi) = \widehat{\mathcal{C}u}(t)$$ i.e. the π-shift and conjugation are interchangable.

Lemma 10.2.4 Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : \bar{D} \to \mathbb{C}$ be such that

(i) $f \in C^{2,\alpha}(\bar{D})$ is holomorphic in D;

(ii) $f|_{\partial D}$ is injective so that $f(\partial D)$ is a simple closed Jordan curve Γ.

Let Ω be the interior of Γ (i.e. the bounded component of $\mathbb{C}|\Gamma$).

Then (a) $\Omega = f(D)$; (b) $f : \bar{D} \to \bar{\Omega}$ is a bijection.

Pf: (a) f is non-constant on $\partial D \Rightarrow f$ is nonconstant on D }
f is holomorphic on D }

$\Rightarrow f$ maps open sets to open sets.

$\Rightarrow \partial(f(D)) \subset f(\partial D) = \Gamma$

since $f(\bar{D})$ is closed, and contains $f(D)$

otherwise if $\exists\, x \in \partial(f(D)) \setminus f(\partial D)$ then $x \in \partial(f(D)) \Rightarrow x \in \overline{f(D)} \Rightarrow x \in f(\bar{D})$

$\Rightarrow x \in f(\bar{D}) \setminus f(\partial D) \Rightarrow x \in \underbrace{f(D)}_{\text{open}} \Rightarrow \exists\, B$ s.t. $x \in B \subset f(D)$ ↯ $x \in \partial(f(D))$

- $f(D) \subset \Omega$

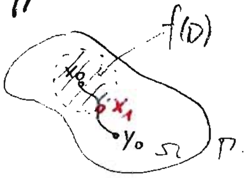
Suppose $f(D) \not\subset \bar{\Omega} \Rightarrow \exists$ a path in $\mathbb{C}|\bar{\Omega}$ joining $x_0 \in f(D)$ to infinity

$\Rightarrow \exists\, x_1 \in \partial(f(D))$ s.t. $x_1 \notin \bar{\Omega} \Rightarrow x_1 \notin \partial\Omega = \Gamma$ ↯ $\partial(f(D)) \subset \Gamma$

thus $f(D) \subset \bar{\Omega}$, but $f(D)$ is open $\Rightarrow f(D) \subset \Omega$.

- $f(D) = \Omega$

suppose $f(D) \neq \Omega$ (since $f(D) \subset \Omega$), $\Rightarrow \exists$ a path in Ω joining $x_0 \in f(D)$ and $y_0 \in \Omega \setminus f(D)$



$\Rightarrow \exists\, x_1 \in \partial(f(D))$ s.t. $x_1 \in \Omega \Rightarrow x_1 \notin \Gamma$ ↯ $\partial(f(D)) \subset \Gamma$.

(b) Let $z_1 \neq z_2 \in \bar{D}$.

- $z_1 \in \partial D,\ z_2 \in D \Rightarrow f(z_1) \neq f(z_2)$

otherwise $f(z_1) = f(z_2) \in f(\partial D) = \Gamma$, i.e. $\exists\, z_2 \in D$ s.t. $f(z_2) \in \Gamma$ ↯ $f(D) = \Omega$ and $\Omega \cap \Gamma = \phi$

- $z_1 \in \partial D,\ z_2 \in \partial D \Rightarrow f(z_1) \neq f(z_2)$ (by assumption)

- $z_1 \in D,\ z_2 \in D \Rightarrow f(z_1) \neq f(z_2)$.

otherwise $f(z_1) = f(z_2) =: \omega \in \Omega$. take $r \in (0,1)$ s.t. $|z_1| < r$, $|z_2| < r$.

$\uparrow f(D) = \Omega$

Set $D_r = \{ z \in \mathbb{C} : |z| < r \}$.

Then $$\frac{1}{2\pi i} \int_{\partial D_r} \frac{f'(z)\,dz}{f(z) - \omega} \geq 2$$

[cf. "Theory of functions" by Titschmarsh pg 116, 3.41,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \#\text{ of zeros of } f \text{ inside } C.$$
]

Also, $|f(z) - \omega|$ is bounded away from 0 on ∂D

$f \in C^{2,\alpha}(\bar{D}) \Rightarrow f' \in L^1(\partial D)$ }

cf. "Harmonic function theory" by Axler et al. $\Longrightarrow$

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)\,dz}{f(z) - \omega} \geq \frac{1}{2\pi i} \int_{\partial D_r} \frac{f'(z)\,dz}{f(z) - \omega} \geq 2$$

$\lightning$

Cauchy's int. formula

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z) - \omega} dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\xi}{\xi - \omega} \overset{\downarrow}{=} 1.$$

Since $\omega \in \Omega$ inside Γ.

$\#$

10.2.4.